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THE JACOBIAN CONJECTURE IN TWO VARIABLES

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The two-dimensional Jacobian conjecture states that given f and g in $\mathbb{C}[x,y]$, if the Jacobian $\partial(f,g)/\partial(x,y)\in\mathbb{C}^\times$, then the transformation $(f,g):\mathbb{C}\times\mathbb{C}\to\mathbb{C}\times\mathbb{C}$ is invertible. In this paper we give a proof of this under the further assumption that the degree of f or g has at most two prime factors.

1. Let f and g be complex polynomials, $f,g\in\mathbb{C}[x,y]$. Put

$$[f,g]=\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix},$$

the Jacobian of $(f,g):\mathbb{C}^2\to\mathbb{C}^2$. If (f,g) is invertible, $x,y\in C[f,g]$, then it is easy to see that $[f,g]$ is a non-zero constant, $[f,g]\in\mathbb{C}^\times$. The Jacobian conjecture is the converse: If $[f,g]\in\mathbb{C}^\times$ then (f,g) is invertible.

2. The Jacobian conjecture is still not settled, but some partial results are known (see [1] for a recent survey). In 1955, A. Magnus [2] proved the following. Put $m=\deg(f)$, $n=\deg(g)$. If $[f,g]\in\mathbb{C}^\times$, and m or n is prime, then (f,g) is invertible. Later, in 1977, Nakai–Baba [3], by making an elegant use of weighted gradings on $\mathbb{C}[x,y]$, extended Magnus' result to include the cases when m or n is 4, and when the larger of m and n is twice an odd prime. Our partial result is the following.

Theorem. *If $[f,g]\in\mathbb{C}^\times$, and m or n has at most two prime factors, then (f,g) is invertible.*

3. Before we get into the proof of this theorem, several remarks are in order. If $(f,g)=(Ax+By^k,Cy)$ or $(f,g)=(Cx,Ay+Bx^k)$, where k is an integer ≥ 0, A and $C\in\mathbb{C}^\times$, and $B\in\mathbb{C}$, then clearly (f,g) is invertible. Such a transformation (f,g) is called an *elementary* transformation. The composite of elementary transformations is clearly invertible. Wright [4] has shown that if (f,g) is invertible, then it is a composite of elementary transformations. Although this is a nice result to know, we do not make use of it. However, in trying to show that a given (f,g) is invertible, we are free to pre- or post-transform it by any elementary transformation.

4. Given f and g in $\mathbb{C}[x,y]$, we shall write $f \sim g$ to mean that $f = Ag$ for some $A \in \mathbb{C}^{\times}$. If f and g are non-zero forms of degrees m and n, respectively, and $[f,g]=0$, then it is easy to see that $f \sim h^{m_1}$ and $g \sim h^{n_1}$ for some form h of degree $d = \gcd(m,n)$, and where $m_1 = m/d$, $n_1 = n/d$ (cf. (22)).

5. Let f and $g \in \mathbb{C}[x,y]$ and suppose $[f,g] \in \mathbb{C}^{\times}$. Let $m = \deg(f)$ and $n = \deg(g)$. If $m = n = 1$, then clearly (f,g) is invertible, $x, y \in \mathbb{C}[f,g]$. Assume that $m > 1$ or $n > 1$. Let f_m (resp. g_n) be the highest homogeneous component of f (resp. g). Then $[f_m, g_n] = 0$ and hence, by (4), $f_m \sim h^{m_1}$ and $g_n \sim h^{n_1}$ for some form h. We may assume that $f_m = h^{m_1}$ and $g_n = h^{n_1}$. If we knew that, say, m divides n, then with $k = n/m$, $\deg(g - f^k) < n$ and we can use induction on the degree to finish the problem. Thus it would be nice to be able to prove: if $[f,g] \in \mathbb{C}^{\times}$, then m divides n or n divides m.

6. In view of (5) we set up the hypothesis as follows. Let $f, g \in \mathbb{C}[x,y]$ and put $\deg(f) = dm$, $\deg(g) = dn$, where $\gcd(m,n) = 1$. Suppose $[f,g] \in \mathbb{C}^{\times}$. We would like to show that $m = 1$ or $n = 1$. Magnus' result is that if $d = 1$, then $m = 1$ or $n = 1$. Nakai–Baba's result is that if $d \leq 2$, then $m = 1$ or $n = 1$. Our result is that if d is a prime number, then $m = 1$ or $n = 1$, i.e. we have the following

Theorem. *Let f, g, m, n and d be as above. If d is* 1 *or a prime number, then $m = 1$ or $n = 1$.*

7. In case it is not completely obvious why the theorem in (6) implies the theorem in (2), here is a proof. Suppose, say, m has at most two prime factors. Then $\gcd(m,n) = 1$, p, q or pq, where p and q are prime numbers. If $\gcd(m,n) = pq$, then $m = pq$ and m divides n. In all other cases, by (6), m divides n or n divides m and n is 1 or a prime number. Then with $k = n/m$ and a suitable $A \in \mathbb{C}^{\times}$, $g_1 = g - Af^k \in \mathbb{C}[f,g]$ has degree $n_1 < n$ (cf. (5)). By induction on $\gcd(m,n)$ such that m or n has at most two prime factors, $x, y \in \mathbb{C}[f, g_1] \subset \mathbb{C}[f,g]$. Similarly for the other case.

8. The rest of this paper is devoted to the proof of the theorem in (6). We shall first list some key lemmas needed to prove the theorem, then deduce the theorem from them, and then turn to the proofs of the lemmas.

9. Following Nakai–Baba we consider various weighted gradings on $\mathbb{C}[x,y]$. By a (rational) *direction* we mean a pair (p,q) of integers such that $\gcd(p,q) = 1$ and $p > 0$ or $q > 0$. Let (p,q) be a direction. A function $f: \mathbb{C}^2 \to \mathbb{C}$ is called *(p,q)-homogeneous* of degree n if

$$f(t^p x, t^q y) = t^n f(x,y) \quad \text{for all } x, y \text{ and } t. \tag{A}$$

10. By a *(p,q)-form* we mean a non-zero (p,q)-homogeneous polynomial $f \in \mathbb{C}[x,y]$;

its (p,q)-degree is denoted by $d_{p,q}(f)$. A (p,q)-form of degree n looks like

$$f=\sum_{pi+qj=n} A_{ij}x^iy^j,$$

where $A_{ij}\in\mathbb{C}$. If $pq<0$, then $d_{p,q}(f)$ could be negative. If f is a (p,q)-form, then every factor of f is a (p,q)-form. Every $f\in\mathbb{C}[x,y]$ has the (p,q)-decomposition $f=\sum_n f_n$ into (p,q)-homogeneous components f_n of degree n. The (p,q) component of highest degree is called the *(p,q)-leading form* of f.

11. Given $f\in\mathbb{C}[x,y]$, let S_f denote the set of points (i,j) in $\mathbb{Z}\times\mathbb{Z}$ such that Ax^iy^j is a term of f for some $A\in\mathbb{C}^\times$. Let W_f denote the convex hull of $S_f\cup\{(0,0)\}$. f is a (p,q)-form iff $S_f\neq\emptyset$ and S_f is contained in a line of slope $-p/q$. We can now state the key lemmas: there are five of them.

12. Lemma. *Let f and $g\in\mathbb{C}[x,y]$ and assume $[f,g]\sim 1$. Put $d_{1,1}(f)=dm$ and $d_{1,1}(g)=dn$, where* $\gcd(m,n)=1$, *$dm>1$ and $dn>1$. Then for each direction (p,q), there is a (p,q)-form h of positive degree such that h^m (resp. h^n) $\sim$ the (p,q)-leading form of f (resp. g).*

13. Lemma. *Let f,g,m,n and d be as in* (12). *Let h be a* $(1,1)$*-form given by* (12). *Then there are inequivalent linear forms x_1,y_1 and distinct non-negative integers a,b such that $h=x_1^ay_1^b$.*

14. Lemma. *Let f, g, m, n and d be as in* (12). *Then there exists a convex polygon (i.e. a closed polygonal region) W with vertices in $\mathbb{Z}\times\mathbb{Z}$ such that $W_f=mW$ and $W_g=nW$.*

15. Lemma. *Let f, g, m, n, d and W be as in* (14). *If E is an edge of W with a negative slope $-p/q$, where* $\gcd(p,q)=1$ *and $q>0$, then $p=1$ or $q=1$.*

16. Lemma. *Let f, g, m, n, d and W be as in* (14). *If d is* 1 *or a prime number, then W is a triangle.*

17. Lemma (12) and the idea behind the proof of Lemma (14) are due to Nakai–Baba. As will be clear from the proof of the theorem given in (19) and (20) below, if we could show that W is a triangle without any condition on d, then the conjecture would be settled.

18. Given f and g as in (14), the convex polygon W is called the *basic web* for (f,g) *in terms* of (x,y). Since $[f,g]\sim 1$, $(1,0)$ and $(0,1)\in S_f\cup S_g$ and hence W must contain vertices $(a,0)$ and $(0,b)$ with $a>0$ and $b>0$.

19. Lemma. *Let f, g, m, n, d and W be as in* (14). *If W is a triangle with vertices* $(0,0)$, $(0,1)$, $(d,0)$, *then $m=1$ or $n=1$.*

Proof. The direction of the edge between $(d,0)$ and $(0,1)$ is $(1,d)$ and $h=y+Ax^d$ with $A\neq 0$ is a $(1,d)$ form given by (12). Put $x_1=x$ and $y_1=y+Ax^d$. We have

$$f\sim y_1^m+\sum_{i+dj<dm} A_{ij}x^iy^j.$$

The term x^iy^j gives rise to terms $\sim x_1^{i+dk}y_1^{j-k}$ for $0\leq k\leq j$ and $(i+dk)+d(j-k)=i+dj<dm$. Thus the basic web W_1 for (f,g) in terms of (x_1,y_1) is a triangle with vertices $(0,1)$, $(0,0)$ and $(d_1,0)$ with $0<d_1<d$, provided $d_1m>1$ and $d_1n>1$. Thus if $m>1$ and $n>1$, then this process can go on forever, which is absurd.

20. Proof of the Theorem. By (16), W is a triangle. We may assume that the vertices of W are $(0,0)$, $(d,0)$ and $(0,c)$ with $c\leq d$. Let (p,q) be the direction of the edge between $(d,0)$ and $(0,c)$. By (15), $p=1$ or $q=1$. Since $c\leq d$, if $q=1$, then $p=1$. Thus $p=1$ in any case, and $cq=d$. if $d=1$ then $c=1$ and we are finished by (19). Suppose d is a prime number and $c>1$. Then $q=1$ and $c=d$. Thus, by (13), there are linear forms $x_1\neq y_1$ and integers $a\neq b$ such that $h=x_1^ay_1^b$ is a $(1,1)$-form as in (12). We have $a+b=d$ and

$$f\sim x_1^{am}y_1^{bm}+\sum_{i+j<dm} A_{ij}x_1^iy_1{}^j.$$

Thus the basic web W_1 for (f,g) in terms of (x_1,y_1) has a vertex (a,b). But, since W_1 is a triangle by (16), $ab=0$, say $b=0$. Then $a=d$ and W_1 has vertices $(d,0)$ and $(0,c_1)$ with $0<c_1<d$. Then $c_1=1$ and we are finished by (19).

21. We now turn to the proofs of the 5 lemmas (12) through (16). These proofs are in turn dependent on various lemmas. First of all, by differentiating the relation (A) with respect to t, we obtain

Euler's Lemma. *If $f:\mathbb{C}^2\to\mathbb{C}$ is a differentiable (p,q)-homogenous function of degree n, then $pxf_x+qyf_y=nf$.*

22. Lemma. *Let f and g be (p,q)-forms of degrees $dm>0$, $dn>0$, respectively, where $\gcd(m,n)=1$. If $[f,q]=0$, then there is a (p,q)-form h of degree d such that $f\sim h^m$ and $g\sim h^n$.*

Proof (cf. Proposition 2 of [3]). First suppose that $m=n=1$. Then by Euler's lemma

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y\end{pmatrix}\begin{pmatrix} px \\ qy\end{pmatrix}=d\begin{pmatrix} f \\ g\end{pmatrix}$$

and hence

$$\begin{pmatrix} g_y & -f_y \\ -g_x & f_x\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}.$$

This implies that $f\sim g$. In case $m\neq n$, by considering f^n and g^m, we get that $f^n\sim g^m$. The result now follows using unique factorization in $\mathbb{C}[x,y]$.

23. Let f and $g \in \mathbb{C}[x,y]$ and suppose $[f,g] \sim 1$. Let $f = \sum f_i$ and $g = \sum g_j$ be the (p,q)-decompositions into (p,q)-forms f_i (resp. g_j) of degree i (resp. j). Then since the (p,q)-degree of $[f_i, g_j]$ is $i+j-p-q$,

$$\sum_{i+j=k} [f_i, g_j] \begin{cases} =0, & \text{if } k \neq p+q, \\ \sim 1, & \text{if } k = p+q. \end{cases} \tag{B}$$

Lemma (12) follows from this and (22).

24. Lemma. *Let f, g, m, n, d and h be as in* (12). *Then there is a (p,q)-form w and an integer $N>0$ such that $[h,w]=h^N$.*

25. Since the proof of (24) requires an entirely different type of argument, we postpone it until the end. Let us assume it for now and prove the remaining lemmas (13) through (16).

26. Lemma. *Let φ be an irreducible (p,q)-form and F a (p,q)-homogeneous rational function of degree $n \neq 0$ belonging to the local ring*

$$R_\varphi = \{ f/g \mid f, g \in \mathbb{C}(x,y], g \not\equiv 0 \ (\mathrm{mod}\ \varphi)\}.$$

If $[\varphi, F] \equiv 0$ (mod φ), then $F \equiv 0$ (mod φ).

Proof. By Euler's lemma and the hypothesis,

$$\begin{pmatrix} \varphi_x & -\varphi_y \\ qy & px \end{pmatrix} \begin{pmatrix} F_y \\ F_x \end{pmatrix} \equiv \begin{pmatrix} 0 \\ nF \end{pmatrix} \ (\mathrm{mod}\ \varphi)$$

and hence, with $d = d_{p,q}(\varphi)$,

$$\begin{pmatrix} px & \varphi_y \\ -qy & \varphi_x \end{pmatrix} \begin{pmatrix} 0 \\ nF \end{pmatrix} \equiv d\varphi \begin{pmatrix} F_y \\ F_x \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ (\mathrm{mod}\ \varphi).$$

Thus $\varphi_y nF \equiv 0$ and $\varphi_x nF \equiv 0$ (mod φ). But, since $\varphi_y n \not\equiv 0$ or $\varphi_x n \not\equiv 0$ (mod φ), $F \equiv 0$ (mod φ).

27. Lemma. *Let w and h be (p,q)-forms with $d_{p,q}(w) \neq 0$. If $[h,w] \equiv 0$ (mod h), then every irreducible factor of h divides w.*

Proof. Let φ be an irreducible factor of h and put $h = \varphi^i h_1$, $h_1 \not\equiv 0$ (mod φ). Then $[h,w] \equiv i\varphi^{i-1} h_1 [\varphi, w]$ (mod φ^i) and hence $[\varphi, w] \equiv 0$ (mod φ). Thus, by (26), φ divides w.

28. Lemma. *Let w and h be (p,q)-forms such that $[h,w] = h^N$ for some $N > 1$. If $p+q>0$ and $d_{p,q}(h) \geq 0$, then h divides w.*

Proof. Let φ be an irreducible factor of h and put $h = \varphi^i h_1$ and $w = \varphi^j w_1$, $h_1 w_1 \not\equiv 0$

(mod φ). If $i<j$, then φ^{i+1} divides w. Assume $i\geq j$. Modulo φ^{2i},

$$\begin{aligned}0&\equiv[h,w]=[\varphi^i h_1,\varphi^j w_1]\\&=\varphi^{i+j}[h_1,w_1]+i\varphi^{i+j-1}h_1[\varphi,w_1]-j\varphi^{i+j-1}w_1[\varphi,h_1]\end{aligned}$$

and hence $ih_1[\varphi,w_1]-jw_1[\varphi,h_1]\equiv 0 \pmod{\varphi}$. Consider $F=w_1^i/h_1^j\in R_\varphi$. We have

$$[\varphi,F]=(w_1^{i-1}/h_1^{j+1})(ih_1[\varphi,w_1]-jw_1[\varphi,h_1])\equiv 0 \pmod{\varphi}.$$

We claim that $d_{p,q}(F)\neq 0$. Put $\alpha=d_{p,q}(h)$, $\beta=d_{p,q}(w)$, $\alpha_1=d_{p,q}(h_1)$, $\beta_1=d_{p,q}(w_1)$ and $d=d_{p,q}(\varphi)$. Since $[h,w]=h^N$, $\beta=(N-1)\alpha+p+q$. Since $N>1$, $\alpha\geq 0$ and $p+q>0$, $\beta>\alpha$. Since $i>0$, $i\geq j$ and $\alpha\geq 0$, $i\beta>j\alpha$. Since $\alpha=id+\alpha_1$ and $\beta=jd+\beta_1$, we get that $i\beta_1>j\alpha_1$. Thus $d_{p,q}(F)=i\alpha_1-j\beta_1>0$. By (26), $F\equiv 0 \pmod{\varphi}$ and hence $w_1\equiv 0 \pmod{\varphi}$ which is false. Thus $i<j$. We have shown that if $\varphi_1,\dots,\varphi_r$ are the irreducible factors of h, then $h\varphi_1\cdots\varphi_r$ divides w.

29. Corollary. *Let f, g, m, n, d and h be as in* (24). *If $p+q>0$, then $[h,w]=h$ for some (p,q)-form w.*

30. Lemma. *Given h and w in $\mathbb{C}[x,y]$, if $[h,w]=h$, then w is square-free.*

Proof. Let φ be an irreducible factor of w and put $h=\varphi^i h_1$ and $w=\varphi^j w_1$ where $h_1w_1\not\equiv 0 \pmod{\varphi}$. Then $\varphi^i h_1=[\varphi^i h_1,\varphi^j w_1]\equiv 0 \pmod{\varphi^{i+j-1}}$ and hence $i\geq i+j-1$ which implies $j\leq 1$.

31. Lemma. *Let h be a $(1,1)$-form. If $[h,w]=h$ for some $(1,1)$-form w, then $h=x_1^a y_1^b$ for some linear forms $x_1 \nparallel y_1$ and non-negative integers $a\neq b$.*

Proof. Since $[h,w]=h$, $d_{1,1}(w)=2$. w is square-free by (30). Thus $w=x_1y_1$ for some linear forms $x_1 \nparallel y_1$. Then, by (27), $h=x_1^a y_1^b$ for some a and b. Since $[h,w]=[x_1^a y_1^b, x_1y_1]=(a-b)x_1^a y_1^b[x_1,y_1]$, $a\neq b$.

32. Lemma (13) now follows from (29) and (31). We can also take care of Lemma (14); it is a consequence of (13). Let $h=x_1^a y_1^b$ be as in (13). W_h is a triangle or a line segment. Let (a_1,b_1) be a vertex of W_h other than $(0,0)$ and drop the subscripts. Then (ma,mb) (resp. (na,nb)) is a vertex of W_f (resp. W_g). Suppose $a>0$. Then let E_f (resp. E_g) be the left edge of W_f (resp. W_g) from (ma,mb) (resp. (na,nb)). Claim that E_f and E_g are parallel. (This is trivial if W_h is a right triangle.) Since $(0,1)\in S_f\cup S_g$, the slope of E_f or of E_g is $<b/a$. Let (i,g) (resp. (k,l)) be the other vertex of E_f (resp. E_g). Suppose E_f and E_g are not parallel. Then

$$\frac{mb-j}{ma-i}\neq\frac{nb-l}{na-k},$$

say $>$. Then $l/k>b/a$ and we can choose a direction (p,q) such that (ma,mb) is the (p,q)-leading point of S_f and (k,l) is the (p,q)-leading point of S_g. Then we get the contradiction that

$$[x^{ma}y^{mb}, x^k y^l] = m(al-bk)x^{ma+k-1}y^{mb+l-1} = 0$$

by (23). Thus E_f and E_g are parallel. Next we claim that $i/j = k/l$. In fact, take a direction (p,q) such that (i,j) (resp. (k,l)) is the (p,q)-leading point of S_f (resp. S_g). Then by (23)

$$[x^i y^j, x^k y^l] = (il-jk)x^{i+k-1}y^{j+l-1} = 0$$

and hence $il = jk$. We now have the similarity of the triangles

$$((0,0),(ma,mb),(i,j)) \quad \text{and} \quad ((0,0),(na,nb),(k,l)).$$

Since $(m,n)=1$, we get $(a_1,b_1) \in \mathbb{Z}\times\mathbb{Z}$ such that

$$(i,j) = (ma_1, mb_1) \quad \text{and} \quad (k,l) = (na_1, nb_1).$$

If $a_1 > 0$, then repeat the argument above. If $b>0$, then we can go to the right also. In this way we arrive at the desired polygon W. This completes the proof of (14).

33. Lemma. *Let h and w be (p,q)-forms such that $[h,w]=h$.*

(i) *If $p>1$ and $q>1$, then $h \sim x^a y^b$ for some non-negative integers $a \neq b$.*

(ii) *If $p=1$ and $q>1$, then $h \sim x^a(y+Bx^q)^b$ for some non-negative integers $a \neq b$ and $B \in \mathbb{C}$.*

Proof. Since $[h,w]=h$, $d_{p,q}(w)=p+q$. If $p>1$ and $q>1$, then $w \sim xy$ and hence $h \sim x^a y^b$ for some non-negative integers a and b by (27) and $a \neq b$ as in (31). Suppose $p=1$ and $q>1$. Then

$$w = Axy + Bx^{q+1}$$

for some A and $B \in \mathbb{C}$. $A \neq 0$ by (30) and $w \sim x(y+Bx^q)$ for some $B \in \mathbb{C}$. Then $h \sim x^a(y+Bx^q)^b$ for some non-negative integers a and b by (27) and $a \neq b$ as before.

34. Lemma (15) follows from (29) and (33(i)). In fact, let h be a (p,q)-form given by (12) for the direction (p,q) of the edge E. Then $p>0$, $q>0$ and h is not a monomial. Thus $p=1$ or $q=1$.

35. Lemma. *Let h and w be $(0,1)$-forms such that $[h,w]=h$. Let $x^a y^b$ be the $(1,1)$-leading term of h and $b>0$. If b does not divide a, then $h=(x-A)^a y^b$ for some $A \in \mathbb{C}$.*

Proof. Put $h = y^b H(x)$, $H \in \mathbb{C}[x]$. Since $[h,w]=h$, $d_{0,1}(w)=1$. Put $w = yW(x)$, $W \in \mathbb{C}[x]$. If $\deg W = 1$, then $w \sim (x-A)y$ for some $A \in \mathbb{C}$ and hence $h=(x-A)^a y^b$ by (27). Suppose $n = \deg W > 1$. We have

$$[y^b H, yW] = y^b(H'W - bHW') = y^b H$$

and hence

$$H'W - bHW' = H.$$

The formal degree of the left side is $a+n-1$ and this is $>a$. Thus $a-bn=0$ and b divides a.

36. Lemma. *Let h and w be (p,q)-forms such that $[h,w]=h$. Let $p<0$, $x^a y^b$ be the $(1,1)$-leading term of h, $a>b>1$ and $-p/q \le (b-1)/a$. If $(a,b)=1$, then $h=x^a y^b$.*

Proof. Replace $-p$ by p so that the direction is $(-p,q)$. Put $h=x^i y^j H(t)$ and $w=x^k y W(t)$, where $t=x^q y^p$ and $H(0)W(0)\neq 0$. Put $\alpha=d_{-p,q}(h)$, $\beta=d_{-p,q}(w)$ and $c=il-jk$; $\alpha=qj-pi=qb-pa$ and $\beta=ql-pk$. Since $[h,w]=h$, $\beta=q-p$ and hence $k=l=1$ by (30). Thus $w=xyW$ and we have

$$[h,w]=x^i y^j(\beta t H'W - \alpha t HW' + cHW) = x^i y^j H.$$

If $c=0$, the left side is divisible by y^{j+p}. Thus $c\neq 0$ and

$$\beta t H'W - \alpha t HW' + cHW = H. \tag{C}$$

Put $m=\deg h$ and $n=\deg W$. If $n=0$, then $m=0$ by (27) and $h=x^i y^j = x^a y^b$. Suppose $n>0$. Since the formal degree of the left side of (C) is $m+n$,

$$\beta m - \alpha n + c = 0. \tag{D}$$

It remains to show that this is impossible under the conditions imposed on (a,b) and $(-p,q)$. We have

$$a=i+mq \quad \text{and} \quad b=j+mp.$$

Since $\alpha=qb-pa$ and $\beta=q-p$ and $c=j-i$, (D) gives that

$$a(pn+1)=b(qn+1).$$

Suppose $(a,b)=1$. Then

$$pn+1=b\lambda \quad \text{and} \quad qn+1=a\lambda$$

for some integer $\lambda>0$. Then

$$\frac{b\lambda-1}{a\lambda-1}=\frac{p}{q}\le\frac{b-1}{a}$$

and hence $a(\lambda-1)+b\le 1$. Thus $b=1$, a contradiction.

37. Let f, g, m, n, d and W be as in (14). Call a vertex (a,b) of W *positive* if there is a direction (p,q) such that $p>0$, $q>0$ and (a,b) is the (p,q)-leading point of W. If (a,b) is a positive vertex of W, then $a\neq b$. In fact, choose a direction (p,q) such that $p>1$, $q>1$ and (a,b) is the (p,q)-leading point of W and let h be a (p,q)-form given by (12). Then by (29) and (33(i)), $a\neq b$.

38. Lemma. *Let f, g, m, n, d and W be as in* (37). *Let (a,b) be a positive vertex of W such that $ab>0$ and E the left edge of W from (a,b). Let (p,q) be the direction of E; note $q>0$. If $p>0$, then $q=1$ and the other vertex of E is $(0,pa+b)$.*

Proof. Let h be a (p,q)-form given by (12). Suppose $p>0$. Then $[h,w]=h$ for some (p,q)-form w by (29). Suppose $q>1$. If $p>1$, then h is a monomial by (33(i)). Thus $p=1$. Then by (33(ii)), $h\sim x^i(y+Ax^q)^j$ for some non-negative integers $i\neq j$ and $A\in\mathbb{C}$. Since h is not a monomial, $A\neq 0$. Since E is the convex hull of S_h, $(a,b)=(i+qj,0)$ and $b=0$. Thus $q=1$. Then by (33(ii)), $h\sim y^j(x+Ay^p)^i$ for some non-negative integers $i\neq j$ and $A\in\mathbb{C}^\times$. Since E is the convex hull of S_h, $(i,j)=(a,b)$ and the other vertex of E is $(0,pa+b)$.

39. Lemma. *Let f, g, m, n, d, W, (a,b) and (p,q) be as in* (38). *If $p\leq 0$, then $b>1$.*

Proof. Suppose $b=1$. Then $p=0$ and the edge E of (38) connects $(a,1)$ and $(0,1)$ and $h=yH$, $H\in\mathbb{C}[x]$. Let x_1 be a linear factor of H and put $H=x_1^cH_1$, $H_1\not\equiv 0 \pmod{x_1}$. Then in terms of x_1 and $y_1=y$,

$$f=y_1^m x_1^{cm} H_1(x_1)^m+\sum_{j<m} A_{ij}x_1^i y_1^j.$$

Thus the basic web W_1 for (f,g) in terms of (x_1,y_1) has $(a,1)$ and $(c,1)$ as vertices. But since W_1 must have a vertex $(0,l)$ with $l>0$, this is impossible. Thus $b>1$.

40. Lemma. *Let f, g, m, n, d, W, (a,b) and (p,q) be as in* (38). *If $a>b>0$ and $(a,b)=1$, then $p>0$.*

Proof. Suppose $p\leq 0$. Then $b>1$ by (39). Let h be a (p,q)-form given by (12). First suppose that $p<0$. Since $a>b>1$ and W has a vertex $(0,c)$ with $c>0$,

$$\frac{-p}{q}\leq\frac{b-c}{a}\leq\frac{b-1}{a}<\frac{b}{a}<1.$$

Since $-p/q<1$, $p+q>0$ and hence, by (29), $[h,w]=h$ for some (p,q)-form w. Then by (36), h is a monomial. Thus $p=0$. Since $b>1$ and $(a,b)=1$, b does not divide a. Thus by (35), $h=(x-A)^a y^b$ for some $A\in\mathbb{C}$. In terms of the variables $x_1=x-A$ and $y_1=y$, we are in the case $p<0$.

41. We can now prove the final lemma (16). In fact, let (a,b) be a $(1,1)$-leading point of W; $a+b=d$. If $d=1$, then W is clearly a triangle. Assume d is a prime number. We may assume that (a,b) is a positive vertex of W, say $a>b$. We want to show that $b=0$. Suppose $b>0$. Then $(a,b)=1$. Let E be the left edge of W from (a,b), (p,q) its direction and h a (p,q)-form given by (12). Then by (40), $p>0$. Then by (38), $q=1$ and $(0,pa+b)$ is the other vertex of E. But since (a,b) is a $(1,1)$-leading point of W, $pa+b\leq d$. Thus $p=1$ and $h=(x+Ay)^a y^b$ for some $A\in\mathbb{C}^\times$ (cf. (38)).

Put $x_1 = x + Ay$ and $y_1 = y$ and consider the basic web W_1 for (f, g) in terms of (x_1, y_1). (a, b) is the $(1, 1)$-leading point of W_1 and is a positive vertex of W_1. Applying the same argument to W_1 we arrive at a contradiction.

42. It remains to prove Lemma (24). We approach this quite formally. Let $t, x_1, x_2, \ldots$ be variables and consider the ring

$$R = \mathbb{C}[t, t^{-1}, x_1, x_2, \ldots].$$

Let α and δ be integers >0 and introduce a grading on R by assigning degrees to t and the x_i's by

$$\deg t = \delta \quad \text{and} \quad \deg x_i = \delta\alpha - i.$$

43. Lemma. *Let β be an integer >0 and put $y_j = 0$ for all $j < 0$ and $y_0 = t^\beta$. Then there exists a family of homogeneous polynomials $y_j \in R$ of degree $\delta\beta - j$ such that*

$$\frac{\partial y_j}{\partial x_i} = \frac{t^{1-\alpha}}{\alpha} \frac{\partial y_{j-i}}{\partial t} \tag{E}$$

for all $i > 0$ and $j \in \mathbb{Z}$.

Proof. Let $j > 0$ and assume we have y_l for all $l < j$. In view of (E), to see y_j exists, it is sufficient to verify that

$$\frac{\partial}{\partial x_k}\left(\frac{\partial y_{j-i}}{\partial t}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial y_{j-k}}{\partial t}\right) \tag{F}$$

for all $i > 0$ and $k > 0$. Since $j - i < j$,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial y_{j-i}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial y_{j-i}}{\partial x_k}\right) = \frac{\partial}{\partial t}\left(\frac{t^{1-\alpha}}{\alpha} \frac{\partial y_{j-i-k}}{\partial t}\right)$$

and this is equal to the right side of (F) by symmetry.

44. Let $\{y_j\}$ be the general solution of (E). Since $y_{j-i} = 0$ for $i > j$, y_j is independent of x_i for $i > j$. By the degree condition, y_j contains a term of the form ct^ν with an arbitrary constant $c \in \mathbb{C}$ iff $j > 0$ and δ divides j, and if so, $\nu = \beta - j/\delta$. If specific numbers are chosen for the arbitrary constants, the resulting family is called a *particular system.*

45. Lemma. *Let $\{y_j\}$ be a particular system and let $\{u_i\}$ and $\{v_j\}$ be families of differentiable functions $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that $u_i = 0$ for $i < 0$ and $u_0 = h^\alpha$. Given $j > 0$, if $v_l = y_l(h, u_1, \ldots, u)$ for all $l < j$, then*

$$\sum_l [u_{j-l}, v_l] = [h, \alpha h^{\alpha-1}(v_j - w_j)], \quad \text{where } w_j = y_j(h, u_1, \ldots, u_j).$$

Proof. Let $'$ indicate the partial derivative with respect to either variable in the domain. We have $u_0' = \alpha h^{\alpha-1} h'$ and

$$v_l' = \frac{\partial y_l}{\partial t} h' + \sum_{i=1}^{l} \frac{\partial y_l}{\partial x_i} u_i'$$

for all $l<j$ with the understanding that the partial derivatives of y_l are evaluated at $(h, u_1, \ldots, u)$. Thus

$$\sum_{l=0}^{j} [u_{j-l}, v_l] = \alpha h^{\alpha-1}[h, v_j] + \sum_{l=0}^{j-1} [u_{j-l}, v_l],$$

and

$$\sum_{l=0}^{j-1} [u_{j-l}, v_l] = \sum_{l=0}^{j-1} \frac{\partial y_l}{\partial t} [u_{j-l}, h] + \sum_{l=1}^{j-1} \sum_{i=1}^{l} \frac{\partial y_l}{\partial x_i} [u_{j-l}, u_i].$$

By using (E) one easily verifies that

$$\sum_{l=0}^{j-1} \frac{\partial y_l}{\partial t} [u_{j-l}, h] = \alpha h^{\alpha-1}[w_j, h]$$

and

$$\sum_{l=1}^{j-1} \sum_{i=1}^{l} \frac{\partial y_l}{\partial x_i} [u_{j-l}, u_i] = 0.$$

Thus we get the equality.

46. Let (p,q) be a direction and h a (p,q)-form of degree δ which is not a proper power. Put

$$D_{p,q} = \{ pi + qj \mid i \geq 0, j \geq 0 \}.$$

Let $u_0 = h^\alpha$ and $v_0 = h^\beta$. Let u_i (resp. v_j) be a (p,q)-form of degree $\delta\alpha - i$ (resp. $\delta\beta - j$) with the agreement that $u_i = 0$ (resp. $v_j = 0$) if $i<0$ (resp. $j<0$) or $\delta\alpha - i \notin D_{p,q}$ (resp. $\delta\beta - j \notin D_{p,q}$).

Lemma. *Let h, u_i and v_j be as above. Given $r>0$, if*

$$\sum_l [u_{j-l}, v_l] = 0 \quad \textit{for all } j<r, \tag{G}$$

then there exists a particular system $\{y_j\}$ such that

$$v_j = y_j(h, u_1, \ldots, u_j) \quad \textit{for all } j<r.$$

Proof. Since $v_0 = h^\beta = y_0(h)$, the claim is true for $j \leq 0$. Let $0<j<r$ and assume it for all $l<j$. Then by (45), $[h, v_j - w_j] = 0$, where $w_j = y_j(h, u_1, \ldots, u_j)$. Suppose $v_j \neq w_j$. If $\nu>0$ is sufficiently large, then $h^\nu(v_j - w_j)$ is a (p,q)-form of degree $\sigma = \delta\nu + \delta\beta - j > 0$ and

$$[h, h^\nu(v_j - w_j)] = 0.$$

Put $\tau=(\delta,\sigma)=(\delta,j)$. Then by (22), there is a (p,q)-form h_1 of degree τ such that

$$h \sim h_1^{\delta/\tau} \quad \text{and} \quad h^{\nu}(v_j - w_j) \sim h_1^{\sigma/\tau}.$$

Since h is not a proper power, $\tau=\sigma$ and δ divides j and we get that $v_j = w_j + ch^{\beta - j/\delta}$ for some $c \in \mathbb{C}$. Absorbing the term $ct^{\beta-j/\delta}$ into y_j, we obtain the claim for j.

47. Lemma. *Let h, u_i and v_j be as in* (46) *and put $r=\delta\alpha+\delta\beta-p-q$. If* (G) *holds and*

$$\sum_l [u_{r-l}, v_l] \sim 1, \tag{H}$$

then there exist a (p,q)-form w and an integer $N>0$ such that

$$[h,w]=h^N.$$

Proof. By (46), $v_j = y_j(h,u_1,\dots,u_j)$ for all $j<r$. Then by (45),

$$[h, \alpha h^{\alpha-1}(v_r - w_r)] \sim 1,$$

where $w_r = y_r(h,u_1,\dots,u_r)$. If $N>0$ is sufficiently large, then $w=h^{N+\alpha-1}(v_r-w_r)$ is a (p,q)-form and $[h,w]\sim h^N$.

48. We can now prove (24). Let

$$f=\sum_i f_i \quad \text{and} \quad g=\sum_j g_j$$

be the (p,q)-decompositions as in (23). Put $u_i = f_{dm-i}$ (resp. $v_j = g_{dn-j}$) with the agreement about $u_i=0$ (resp. $v_j=0$) as in (46). Then the conditions (B) become the conditions (G) and (H) and we have $u_0 \sim h^m$ and $v_0 \sim h^n$. We may assume that $u_0=h^m$ and $v_0=h^n$. h may be a proper power; put $h=h_1^e$, where h_1 is not a proper power. Put $\delta=d_{p,q}(h_1)$, $\alpha=em$ and $\beta=en$. Then $d=e\delta$, $\delta\alpha=dm$ and $\delta\beta=dn$. By (47) there exist a (p,q)-form w and an integer $N_1>0$ such that $[h_1,w]=h_1^{N_1}$. Since $[h_1,h_1w]=h_1^{N_1+1}$, we may assume that $N_1 \equiv 1 \pmod e$. Then

$$[h,w]=eh_1^{e-1}[h_1,w] \sim h_1^{e-1+N_1}=h^N,$$

where $N=1+(N_1-1)/e$. This proves (24) and the proof of the theorem is now complete.

49. We conclude the paper with some remarks. Let f, g, m, n, d and W be as in (14). When d is a prime number we get that $m=1$ or $n=1$ by looking at the edges of W (other than the horizontal and vertical edges). Even if d is not a prime number, if d is small, we get that $m=1$ or $n=1$ by looking at the edges of W. Such is the case for $d=4$, 6 and 8; in Lemma (36) even if $(a,b)\neq 1$, if $a+b\leq 8$, then we get that $h=x^a y^b$.

50. But when $d=9$ we can no longer get that $m=1$ or $n=1$ by just looking at the edges of W. In fact, if $(a,b)=(6,3)$, then there are $(-1,3)$-forms h and w such that $[h,w]=h$ and h is not a monomial. Also there are $(1,-1)$-forms h and w such that $[h,w]=h$ and h is not a monomial. This means that in order to get $m=1$ or $n=1$ we must dig deeper into the interior points of S_f and S_g (i.e., the interior points of mW and nW).

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